The University of Nottingham Ningbo China

FHSS School of Economics

ECON3070 Numerical Methods in Economics

Coursework

<Review of a Dynamic Model of Housing Demand: Estimation and Policy Implications>

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Abstract:
Based on U.S. background of economic recession, this paper investigates the problem of household housing demand facing income and house price shocks. Two stage method by Bajari et al. (2013) is used and discrete state dynamic programming and fmin research are numerical methods applied.

Key words: housing demand, house price shock, income shock, BBL method, dynamic programming, fmin research.

1. Introduction

According to Bajari et al. (2013), during economy recession in U.S., house price dropped sharply accompaning by household income decrease. Aiming to mimic real world situation, the economic problem in this paper is how are household housing demand, non-housing financial investment and consumption decisions facing the shocks of house price and income in order to maximize their lifelong utility?

However, considering special properties of the structural model depicting housing market, numerical methods are needed to solve the problem rather than analytical analysis for following reasons. **Firstly,** nonconvex adjustment costs containing house purchase price and transaction cost to change house size are included in the lifelong expected utility function, creating difficulties to perform first order condition and to get a clear maximization result. **Besides**, house size in this question is modeled as a mixed discrete and continuous problem as whether to purchase or rent a house is discrete while the house size is a continuous variable. **Moreover**, the constraint conditions including budget and mortgage collateral constraints are complex.

2. Method

2.1 Two-stage Method Proposed by Bajari et al. (2013)

The two-stage method originated by Bajari et al. (2013) is used to mimic real world and simulate a typical household's optimized decision in housing market. One crucial advantage is this method can solve the realistic problem without directly addressing dynamic programming problem thus it avoids repeat optimal decisions computation with real-life data. In the first stage, discrete state dynamic programming as a numerical tool is applied to estimate the optimized general formulations of housing, consumption and investment decisions. In the second stage, parameters in the decision rule functions developed previously are estimated using fmin research with two alternative estimation methods mentioned later to provide consistent estimators.

2.2 The First Stage

2.2.1 Theoretical Framework of Discrete State Space Dynamic programming (DP)

a) **Discrete-time, discrete-space Markov decision model**

In Markov process, agents' rewards depend both on the state of process and the action taken (Howard, 1960). Furthermore, assuming the distribution of current state action only conditional on the last period's action and state due to the memoryless property, housing decision modelled by Markov process implies household choice at current state only depends on housing value and financial investment at previous stage.

b) Principle of Optimality

In dynamic programming, instead of optimizing all T at once, the optimization is conducted on one period at time (Bellman, 1965). Based on Principle of Optimality, the remaining decisions must constitute an optimal policy regardless of initial states and previous actions. Therefore, assuming the value at terminal period T+1 is a specific salvage value, the finite horizon decision model can be redressed through backward substitution. Additionally, since Action Space (X) containing all agents' possible actions and State Space (S) including all process states are both finite, by substituting recursively and solving Bellman's equation repeatedly through performing arithmetic operations, the optimization problem can be redressed.

2.2.2 Application of Discrete DPs on the First-state Estimation

Following the spirit of BBL (2007), this section would firstly simulate the exogenous state variable (income, home price and interest rates) governed by the law of motion to obtain α_i and p_t, before introducing the housing investment policy function (ordered Probit model), bequest function and expected utility function to establish Bellman Equation. After basic model construction, this paper would demonstrate how to solve the dynamic programming using value function iteration.

2.2.3 Modelling Discrete DPs

a) Exogenous State Variable

Income

To forward simulate the income process, Bajara et,al. (2013) include an age component, a cohort effect[1], household random effects, and an AR (1) error term, formulating as following:

$$\alpha_i = \log(y_{it}) - \left(\beta_0 + \beta_1 \text{ age }_{it} + \beta_2 \text{ age }_{it}^2 + \beta_3 \text{ birthcohort }_i\right) \tag{1}$$

Home Price Inflation and Interest Rate

[1] Cohort effect: Income: a commonly aged group of people may impose indirect effect on research due to their common-age related influences.

Real interest rate (r) and home price inflation are modelled as a VAR with one lag with the initial value of year 1 extracting from the dataset.

$$r_t = \beta_{cr} + \beta_{rr} r_{t-1} + \beta_{r\pi} \pi_{t-1} + e_{rt}$$
$$\pi_t = \beta_{c\pi} + \beta_{\pi\pi} \pi_{t-1} + \beta_{\pi r} r_{t-1} + e_{\pi t}$$

$$P_t = (1+\pi_t)^* \ P_1 \tag{2}$$

b) Transformation of the State Space

Housing Investment Policy Function

$$p_{it,j} = P[h_{it} \in j] = \Phi\left(\alpha_j - F(\mathbf{s}_{it};\beta)\right) - \Phi\left(\alpha_{j-1} - F(\mathbf{s}_{it};\beta)\right) \tag{3}$$

To construct the transition probability function for housing investment, ordered Probit model is adopted. Equivalently, under one of four categories of choice (rent, downgrade, remain in existing home, upgrade), household would only switch between choice if the associating value of flexible function F(·) surpasses one of three cutoff point values.

Dynamic Decision of Financial Asset

Except housing investment decision (h'), household may also decide the share of capital (a') to devote into financial asset. However, these two choices are not independent, thus to address the additional constraint on h' imposed by a', a new voluntary equity variable (q') is constructed. This enables the construction of rectangular constraint set for (c, q', h').

$$q = a + (1-\xi)p_{-1}h \tag{4}$$

c) The Terminal Value Function and The Discount Expected Utility Function

To conduct grid search for each (q', h') over finite admissible options, the terminal value function needs to be firstly defined. The motivation to leave some assets at the end of period (t=70) associates with either the bequest motives or incentive to save for retirement, which can be specified by bequest function as follows:

$$b_{iT} = max\{q_{iT}, 100\}$$

Household obtains utility from the non-durable consumption and housing service, thus period utility function can be constructed as following for house owners and house renters respectively: The parameter θ captures the share of consumption, τ captures the elasticity of substitution between housing and non-housing consumption, κ captures the utility flow of housing parameter with mean μ_κ and variance σ_κ^2.

If owner:

$$u(c, h' > 0, l = 0) = \log\left[(\theta c^\tau + (1-\theta)(\kappa h')^\tau)^{\frac{1}{\tau}}\right] \tag{5}$$

If renter:

$$u(c, h' = 0, l > 0) = \log\left[(\theta c^\tau + (1-\theta)(l)^\tau)^{\frac{1}{\tau}}\right] \tag{6}$$

For a finite horizontal discrete Markov Chain, the discounted expected utility derives from the summation of period utility function $u(c_t, g^* s_t)$ constituted by immediate reward (current housing, non-durable consumption), expected future rewards (future housing, financial investment, and bequest).

$$U(\{c_t, h_{t+1}, l_t\}_{t=0}^T) = E_0\left[\sum_{t=1}^{T} \beta^{t-1} u(c_t, g^* s_t) + \gamma\beta^T \log(b_T)\right]$$

2.2.4 Solving Discrete DPs by Bellman Equation

To solve for the life-cycle optimal policy, this paper would follow the Principal of Optimality to repeatedly solve the Bellman Equation formulated as following equation, which contains the following components : transition probability (P) based on policy investment policy function (formula 3), period utility function $u(\cdot)\ from\ (formula\ 5\ and\ 6)$, the current ($p$)and next stage ($p'$) house price from (formula 2), the voluntary equality variable (q') from (formula 4), a deterministic term (η').

$$v(\eta, q, h, p_{-1}, p, t) = \max_{c, q', h'}\{u(c, g(d)) + \beta E v(\eta', q', h', p, p', t+1)\} \tag{7}$$

Backward substitution allows the state-contingent actions to yield maximum expected steam of rewards. Having $v(t+1)$, $v(t)$ for all state S can be calculated. Sequentially, $v(t-1)$, $v(t-2)$,…, $v(1)$ can be obtained. After computing all possible input value and exit value associate with q' from period T to the first period in the first loop, h' can be directly addressed through forward substitution in the second loop.

2.2.5 Computing Consumption by Budgets Constraints

Finally, given income computed from equation (1) and optimal (q', h') in each state obtained from (7), the consumption (c_t) can be implied by budgets constraints, indicating as following:

$$c + a' + ph' + p\phi(h', h) = \eta\varepsilon_t + (1 + r(a))a + ph$$

2.3 The Second Stage

Firstly, in this stage, lifelong housing and consumption decisions under policy functions developed before are simulated to fit in real world situation, where a sample set of 450 households of family size 2 is selected to simulate these decisions.

After this, considering the drawback that only partial and static view of controlled dynamic model is involved before, parameters are changed to find optimal ones which generate highest utility under previous decision rules. Equally, parameters are investigated by finding ones corresponding to equilibrium conditions of this model which is shown below based on previously estimated decision rules (Bajari, Benkard and Levin, 2007).

$$V_i(s;\ \sigma_i;\ \sigma_{-i};\ \theta) \geq V_i(s;\ \sigma_i';\ \sigma_{-i};\ \theta)$$

where s captures all states, i denotes different household, σ captures the equilibrium decision set while σ' captures all available decision sets, θ captures parameter set with minimum distance to equilibrium.

There are two alternative parameter estimating methods can be applied including estimation of identified models and bounds estimation, and both target to minimize the distance to equilibrium (Bajari, Benkard and Levin, 2007).

Regarding estimation of identified models, function g(.) is firstly defined to express the gap to equilibrium and a subset of equilibrium inequalities is chosen within the whole set of inequalities with changing parameters. Then, another function Q(.) is defined to combine the previous attempts which is shown below:

$$Q_n(\theta,\alpha):=\frac{1}{n_I}\sum_{k=1}^{n_I}(min\{\hat{g}(X_k;\ \theta,\alpha),0\})^2$$

where X captures a subset of equilibrium inequalities chosen, n_I captures the total number of inequalities selected, θ captures parameter set estimated, α captures value function.

Additionally, bounds estimation method proposed by Romano and Shaikh (2008) is another estimating tool where the initial bound of parameter set is firstly ensured to be large enough to cover the optimal estimators and then the critical values of each subset are computed and compared with the one corresponding to 95% confidence, aiming to contract bound until achieving the 95% confidence requirement. The estimations are regarded as consistent estimators of true value.

The simplex-based fmin research in MATLAB is used to minimize the distance to equilibrium. More specific numerical method under the fmin research hierarchy is selected considering specific estimation method, special properties of data and certain constraints of model.

3. Pseudo-codes

Algorithm Discrete-time discrete space dynamic programing algorithm

1: Begin

2: Set parameters of the problem $(\beta,\ \gamma, g, \sigma, \theta, \tau, e^{\kappa}, \phi, \xi, r, r_m, \rho, \pi_p, \sigma_p, \pi_\eta, \sigma_\eta, \varepsilon_t, T)$

3: Assign initial value to income (y_{i1}), house price (P_1), interest rate (r_1) and home price inflation (π_1) from dataset

4: Model income process using AR (1) to simulate income in each state (y_{it})

$$\alpha_i = \log(y_{it}) - \left(\beta_0 + \beta_1 \text{ age }_{it} + \beta_2 \text{ age }_{it}^2 + \beta_3 \text{ birthcohort }_i\right)$$

5: Set interest rate and price inflation using VAR (1) with one lag to obtain interest rate (r_t) and home price inflation (π_t) in each state

$$r_t = \beta_{cr} + \beta_{rr} r_{t-1} + \beta_{r\pi}\pi_{t-1} + e_{rt}$$
$$\pi_t = \beta_{c\pi} + \beta_{\pi\pi}\pi_{t-1} + \beta_{\pi r} r_{t-1} + e_{\pi t}$$

6: set home price in each state

$$\mathrm{P}_t = (1+\pi_t) * \mathrm{P}_1$$

7: Set transition probability matrix and set α_j choice based on four cases ($\Delta h_{it} < 0, \Delta h_{it} > 0,\ \Delta h_{it} = 0,\ h_{i(t+1)} = 0$)

$$p_{it,j} = P[h_{it} \in j] = \Phi\left(\alpha_j - F(\mathbf{s}_{it};\beta)\right) - \Phi\left(\alpha_{j-1} - F(\mathbf{s}_{it};\beta)\right)$$

8: Set $g(d)$ and the period utility function

If $h_{t+1} > 0$ **then**

$$u\big(c, \mathrm{g}(d)\big) = \log\left[(\theta c^{\tau} + (1-\theta)(\kappa h')^{\tau})^{\frac{1}{\tau}}\right]$$

Else

$$u\big(c, \mathrm{g}(d)\big) = \log\left[(\theta c^{\tau} + (1-\theta)(l)^{\tau})^{\frac{1}{\tau}}\right]$$

9: Set a new voluntary equity variable (q_{it})

$$q_{it} = a_{it} + (1-\xi)p_{t-1}h_{it}$$

10: Set bequest function to obtain (b_{iT}) with dataset to decide initial value of bequest (b_1)

$$b_{iT} = max\{q_{iT}, 100\}$$

11: Set the initial value of house equity equals to initial bequest

$$q_1 = b_1$$

12: Set grid for house equity

$$g = 0{:}\,grid{:}\,g_1$$

13: Set matrix to store value

$$v = [NaN(length(q), T), zeros(length(q), 1)]$$

14: Loop over possible value of q_t and q_{t+1} to maximize the Bellman equation

$$v_t(\eta_t, q_t, h_t, p_{t-1}, p_t) = \max_{c_t, q_{t+1}, h_{t+1}} \left\{u\big(c_t, g_t(d)\big) + p_{it,j}\beta v(\eta_t, q_{t+1}, h_{t+1}, p_t, p_{t+1})\right\}$$

15: Find corresponding q_{t+1} and h_{t+1} in matrix

16: compute c_t

$$c_t + a_{t+1} + p_t h_{t+1} + p_t\phi(h_{t+1}, h_t) = \eta\varepsilon_t + (1 + r_t(a))a_t + p_t h_t$$

17: End

4. Conclusions

Discrete state dynamic programming complemented with fmin research are used to solve the economic problem in this paper. However, the dynamic programming method has limitations. Firstly, no standard performing mode exists as considerations about the subproblems' content, the computation way and the subproblems' order are needed according to specific questions to improve efficiency (Bhowmilk, 2010). In this case, the special properties of housing market and related factors are required to consider in order to write specialized value function. Besides, according to Rust (2008), time consistency is the compulsory precondition to directly use the principle of optimality while in this question some housing decisions which are not directly related to the adjacent periods are ignored. Additionally, only a small number of sub issues can be processed efficiently at one time due to computer storage and speed limitations (Bhowmilk, 2010). Therefore, more detailed kinds of non-durable goods and house consumption cannot be analyzed in this issue.

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16: compute c_t

$$c_t + a_{t+1} + p_t h_{t+1} + p_t\phi(h_{t+1}, h_t) = \eta\varepsilon_t + (1 + r_t(a))a_t + p_t h_t$$

17: End

Codes

```
% BackInduct.m

% Parameters: preferences
beta = 0.97 ;         % Time discount factor
gamma = 2.56 ;        % Degree of alturism
sigma= 1 ;            % Intertemporal elasticity of subset
g  = 7.24 ;           % Service flow from housing stock
theta = 0.539;        % consumption share in utility function
tau  = 0.782;         % elasticity of sub. between c,h
e.^kappa = 0.31:1.28; % domain of the preference shocks (check)

% parameters: housing and Financial Markets
phi = 0.06;       % Transaction cost
xi = 0.2;         % Down-payment requirement
r = 0.01 ;        % Return on financial assets
rm = 0.0724;        % Mortgage interest rate
rho = 0.0724;        % Rental rate of housing
pirho = 0.95;        % Persistence of house price shock
sigmarho = 0.1;       % Std. dev. of house price shock


% parameters: labor income process
pieta = 0.95          % Persistence of income shock
sigmaeta = 0.3        % Std. dev. of income shock

% set the variables to their values in the dataset
household_data = readmatrix('data/.csv');
o1 = household_data("rent/own");
h1 = household_data("house_size");
l1 = household_data("rent_size");
p1 = household_data("mortgage_principal");
a1 = household_data("financial_asset");

%%% Update exogenous state variables (income, home prices, interest rates).


% income process: a random effects model with AR(1) error term, for the
% simulation purpose, transform the random effects into imputed fixed
% effects.
tdist = struct('Name','at','DoF',69);
Mdl = arima(1,0,0)


%simulate interest rate (rt) and home price inflation (piet) as VAR with one
%lag to obtain pt
r_init = household_data("initial_interest_rate")
piet_init = 0;
```

```
numseries = 2;
p =1;
Mdl = varm(numseries,p);
seriesnames = {'rt','piet'};
pt= (1+piet)*p;


% update home equity choice due to home price change and nonhousing wealth
% due to interest accumulation to obtain qt
qt = at +0.8*ht*p(t-1)


% construct the transition probability matrix to obtain P
P = [];
    P = sparse(zeros(4,T));
    for j= 1:3
    P(alpha(1), 0) = cdf(alpha(1) -F(s(it);beta))-cdf(0-F(s(it);beta));
    P(alpha(2), alpha(1)) = cdf(alpha(2) -F(s(it);beta))-cdf(alpha(1)-F(s(it);beta));
    P(alpha(3), alpha(2)) = cdf(alpha(3) -F(s(it);beta))-cdf(alpha(2)-F(s(it);beta));
    P(1, alpha(3)) = cdf(1 -f(ik))-cdf(alpha(3)-F(s(it);beta));
    end
    P = [P;Pk];

%%% Solving Discrete DPs by Bellman Equation
% enter the model parameters and construct the state and action spaces


% define the period utility function for owner and renter respectively
if h(t+1)>0
    u(c,g(d))=log[(theta*c^(tau)+(1-theta)(kappah*h')^(tau))^((1)/(tau))];
else
    u(c,g(d))=log[(theta*c^(tau)+(1-theta)(1)^(tau))^((1)/(tau))];
end

% construct bellman equation and rectangular grid for each period to search
% for optimal ï¼ˆq'ï¼Œh'ï¼‰
for t= 70:-1:1
    for inq=1:length(Q)
        for outq=1:(inq)
            h=Q(inq)-Q(outq);
            V2(inq, outq, t)=u(c, g(d)) + P*beta*V(outq, eta', q', h', p, p', t+1);
        end
    end
    V(:,t)=max(V2(:,:,t),[],2);
end

% Calculate optimal results forward
vf = NaN(T,1);
equity = [q1; NaN(T,1)];
```

```
house = NaN(T,1);
for t = 1:T
    vf(t)=V(find(Q==equity(t))),t);
    equity(t+1)=Q(find(V2(q==cap(t)),:,t)==vf(t)));
    house(t)=equity(t)-equity(t+1);
end

% Display and plot results
disp('   q      h')
fprintf('%3.3f %3.3f\n', equity([1:t], :), house)

subplot(2,1,1)
plot([1:1:T], [con, cap([2:T+1], :)], 'LineWidth', 2)
ylabel('House, Equity', 'FontSize', 12)
xlabel ('Time', 'FontSize', 12)
legend('House', 'Equity')

subplot(2,1,2)
plot([1:1:T], vf, 'Color', 'red', 'LineWidth', 2)
ylabel('Value Function', 'FontSize', 12)
xlabel('Time', 'FontSize', 12)

% compute consumption through budgets constraint
c(t)+a(t+1)+p(t)*h(t+1)+p(t)*phi(h(t+1),h(t))= eta*epsi(t)+(1+r(t)(a))*a(t)+p(t)*h(t)
```